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Transition Probabilities and Stochastic Equations for the Mean Field Ising Model

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The microscopic transition rate is briefly calculated from quantum principles to derive the microscopic master equation. By introducing τ_p, the phenomenological time, and coarse graining W_p, the transition rate, a complete normalized phenomenological transition rate is obtained. The Langer form is then approximately obtained.

KEY WORDS: Stochastic dynamics; mean field Ising model; Langer's transition rate.

1. INTRODUCTION

The phenomenological (mesoscopic) master equation proposed by Langer[1] is based on the assumption that the transition probability is determined by the change of a free energy of the system caused by the transition and a Gaussian factor of the state variables. The latter has the effect of reducing the probability for large changes in the variables. The equation has been successfully used to discuss the decay of the metastable states and spinodal decomposition.[1,2] It is conjectured that the theory should apply to any system whenever the free energy mentioned above can be meaningfully defined.

Metiu *et al.*[3] presented microscopic arguments for the establishment of the phenomenological master equation. Using Zwanzig's projection methods, they derived a microscopic master equation for the mean field

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spin system. Then, they made an additional assumption that a Markoffian master equation exists on a phenomenological time scale, and obtained, by a further averaging, a phenomenological transition rate of Langer's form by means of the path integral method. They concluded that on the phenomenological time scale the stochastic time dependence of the system is determined by its coarse-grained "thermodynamic" properties. One then can ignore the detailed dynamics of the system and heat bath.

A considerable literature exists on the realm of validity of the Pauli equation[4] for open systems and on the manner in which the transition probabilities might be derived from quantum principles.[5] We here, using these results, briefly calculate the microscopic transition probabilities and obtain the microscopic master equation. In Section 3 we derive complete normalized phenomenological transition probabilities. In Section 4 we draw an analogy between spin flips and random walks, and discuss Langer's form as an approximation of the complete phenomenological transition rate.

2. THE MICROSCOPIC MASTER EQUATION

We consider here the mean field Ising spin system[6] coupled to a phonon heat bath[3] in an external magnetic field. The Hamiltonian of the spin–phonon system is

$$H = H_S + H_B + V \tag{1}$$

where H_S, H_B, and V are the Hamiltonian for spins, for phonons, and for the interaction between spins and phonon "heat bath," respectively. In terms of the creation and annihilation operators for up-spin at site $i(\eta_i^+, \eta_i)$ and those for phonon of the mode α with frequency $\Omega_\alpha(O_\alpha^+, O_\alpha)$, we have

$$H_S = -\frac{J}{2N}\left[2\sum_i \eta_i^+\eta_i - N\right]^2 - \mu_B H\left[2\sum_i \eta_i^+\eta_i - N\right] \tag{2a}$$

$$H_B = \sum_\alpha \hbar\Omega_\alpha O_\alpha^+ O_\alpha, \tag{2b}$$

$$V = \sum_{i=1}^{N} \sum_\alpha (\eta_i^+ + \eta_i)(O_\alpha^+ + O_\alpha) g_\alpha, \tag{2c}$$

where J is the coupling constant for spin flips, g_α is that for spin–phonon interaction, N the number of spins, μ_B the Bohr magnetic constant, and H the external magnetic field.

The transition probabilities of spin flips induced by interaction, V, per unit time can be calculated with[5]

$$W = \frac{2\pi}{\hbar} |\langle f|V|i\rangle|^2 \delta(E_f^0 - E_i^0) \tag{3}$$

where $|i\rangle$ and $|f\rangle$, E_i^0 and E_f^0 are the initial and final states and their energies.

Denote by N_+ the eigenvalue of $\sum_i \eta_i^+ \eta_i$, $N_- = N - N_+$, and $M = N_+ - N_-$. From Eqs. (2a) and (2b) we obtain the energy difference for the elementary process $N_+ \to N_+ - 1$

$$E_f^0 - E_i^0 = \frac{J}{N} M + \mu_B H - \hbar\Omega_\alpha - \frac{J}{2N} \equiv \hbar[\omega_-(M) - \Omega_\alpha] \tag{4}$$

for example.

Assuming the phonon heat bath is in the equilibrium state,[5] we obtain

$$\begin{aligned} W(N_+ \to N_+ - 1) &= \frac{2\pi N_+}{\hbar^2} \int d\Omega_\alpha g_\alpha^2 \rho(\Omega_\alpha)\big[\bar{n}_\alpha \delta(w_- - \Omega_\alpha) \\ &\qquad\qquad + (\bar{n}_\alpha + 1)\delta(\omega_- + \Omega_\alpha)\big] \\ &= \frac{2\pi N_+}{\hbar^2} \frac{g_\alpha^2 \rho(|\omega_-|)}{\sinh(\beta\hbar|\omega_-|/2)} \exp\left(-\frac{\beta\hbar\omega_-}{2}\right) \end{aligned} \tag{5}$$

where $\bar{n}_\alpha$ is the number of phonons in mode α. Similarly, we have

$$W(N_+ \to N_+ + 1) = \frac{2\pi N_-}{\hbar^2} \frac{g_\alpha^2 \rho(|\omega_+|)}{\sinh(\beta\hbar|\omega_+|/2)} \exp\left(-\frac{\beta\hbar\omega_+}{2}\right) \tag{6}$$

where

$$\hbar\omega_+ = -\frac{J}{N} M - \mu_B H - \frac{J}{2N} \tag{7}$$

By introducing the "microscopic" free energy for the spin system

$$F(N_+) = \frac{J}{2N} M^2 + \mu_B HM - \beta^{-1} \frac{N!}{N_+! N_-!} \tag{8}$$

to order $1/N$, Eqs. (5) and (6) can be written as

$$W(N_+ \to N_+ \pm 1) = \frac{2\pi (N_+ N_-)^{1/2}}{\hbar^2 \sinh(\beta\hbar\omega/2)} \exp\left(\mp \frac{\beta \Delta F}{2}\right) \tag{9}$$

where

$$\hbar\omega = \frac{J}{N} M + \mu_B H$$

$$\Delta F = \hbar\omega - \beta^{-1} \ln\left(\frac{N_-}{N_+}\right)$$

By means of $W(N_+ \to N_+ \pm 1)$ we can then write the master equation[5] as

$$\frac{\partial}{\partial t} P(N_+, t) = W(N_+ + 1 \to N_+) P(N_+ + 1, t) + W(N_+ - 1 \to N_+) P(N_+ - 1, t) - [W(N_+ \to N_+ + 1) + W(N_+ \to N_+ - 1)] P(N_+, t) \tag{10}$$

This is the same result as obtained by Metiu *et al.*[3]

3. THE PHENOMENOLOGICAL TRANSITION PROBABILITIES

Equation (10) may be rewritten as

$$\frac{\partial \mathbf{P}(t)}{\partial t} = \hat{\mathcal{K}} \mathbf{P}(t) \tag{11}$$

The nonvanishing elements of Matrix $\hat{\mathcal{K}}$ are

$$\begin{aligned} \mathcal{K}_{N_+, N_+ \pm 1} &= W(N_+ \pm 1 \to N_+) \\ \mathcal{K}_{N_+, N_+} &= -[W(N_+ \to N_+ + 1) + W(N_+ \to N_+ - 1)] \end{aligned} \tag{12}$$

For a given initial condition $\mathbf{P}(0)$ the solution to Eq. (11) can be written in the following form:

$$\begin{aligned} \mathbf{P}(t) &= \mathbf{P}(0) + \int_0^t dt_1 \hat{\mathcal{K}} \mathbf{P}(0) + \int_0^t dt_1 \hat{\mathcal{K}} \int_0^{t_1} dt_2 \mathcal{K} \mathbf{P}(0) + \cdots \\ &\equiv \hat{\mathcal{O}}(t) \cdot \mathbf{P}(0) \end{aligned} \tag{13}$$

The phenomenological transition rate $\hat{W}^P$ is defined by

$$\mathbf{P}(\tau_p) = \hat{W}^P \mathbf{P}(0)$$

Thus,

$$P_n(\tau_p) = \sum_m W_P(m \to n) P_m(0) \equiv \sum_m W_{nm}^P P_m(0) \tag{14}$$

where τ_p is the phenomenological time scale. Therefore,

$$\hat{W}^P = \hat{\mathcal{O}}(\tau_p) \tag{15}$$

The phenomenological coarse time scale τ_p introduced by Metiu *et al.*[3] is larger than the microscopic time scale τ_m, which can properly be defined as the average time for one spin flip,

$$\tau_m = [W(N_+ \to N_+ + 1) + W(N_+ \to N_+ - 1)]^{-1} \tag{16}$$

If τ_p is still sufficiently small so that the change of magnetization on that time scale is small, then we may adopt the approximation[3]

$$W(N_+ \to N_+ \pm 1) = \bar{\alpha} \exp(\mp \bar{\beta}) \tag{17}$$

where $\bar{\alpha}$ is the mean value of

$$\alpha(N_+) \equiv \frac{2\pi (N_+ N_-)^{1/2} \rho(\omega)}{\hbar^2 \sinh(\beta \hbar \omega / 2)}$$

in time interval τ_p, and $\bar{\beta}$ the mean of $\beta \Delta F/2$. We shall use Eq. (17) to calculate $\hat{W}^P$. From Eqs. (13) and (15) $\hat{W}^P$ is then a sum of terms whose general form is

$$\int_0^{\tau_p} dt_1 \int_0^{t_1} dt_2 \cdots \int_0^{t_{\nu_p - 1}} (\hat{\mathcal{H}})^{\nu_p} = \frac{(\tau_p)^{1/p}}{\nu_p!} (\hat{\mathcal{H}})^{\nu_p} \tag{18}$$

Collecting all the terms contributing to $\langle n + \nu | \hat{W}^P | n \rangle$ of the transition matrix $\hat{W}^P$, we have

$$W_p(n \to n + \nu) = \sum_{\nu_p} \sum_{\nu_0}{}' \frac{(\bar{\alpha}\tau_p)^{\nu_p}}{\nu_p!} \frac{\nu_p!}{\nu_+! \, \nu_-! \, \nu_0!} \left(-2\cosh \bar{\beta}\right)^{\nu_0} e^{-\nu \bar{\beta}} \tag{19}$$

where the prime in $\sum'_{\nu_0}$ means the restriction

$$\begin{aligned} \nu_+ + \nu_- + \nu_0 &= \nu_p \\ \nu_+ - \nu_- &= \nu \end{aligned} \tag{20}$$

Changing the order of the two summations in Eq. (19), we obtain

$$\begin{aligned} W_p(n \to n + \nu) &= \sum_{\nu_0 = 0}^{\infty} \frac{\left(-2\bar{\alpha}\tau_p \cosh \bar{\beta}\right)^{\nu_0}}{\nu_0!} \sum_{\nu_- = 0}^{\infty} \frac{(\bar{\alpha}\tau_p)^{2\nu_- + \nu}}{(\nu_- + \nu)! \, \nu_-!} e^{-\nu \bar{\beta}} \\ &= e^{-2\bar{\alpha}\tau_p \cosh \bar{\beta}} I_\nu (2\bar{\alpha}\tau_p) e^{-\nu \bar{\beta}} \end{aligned} \tag{21}$$

where we have used Eq. (20) and the series for modified Bessel functions $I_\nu(x)$[7]

$$I_\nu(x) = \sum_{S=0}^{\infty} \frac{1}{s! \, (s + \nu)!} \left(\frac{x}{2}\right)^{2s + \nu} \tag{22}$$

4. RELATION OF SPIN FLIPS TO RANDOM WALKS

The model is closely related to one-dimensional random walk with continuous time.[8,9] The master equation for one-dimensional random walk is

$$\frac{\partial P(t,s)}{\partial t} = kP(t,s-1) - (k+k')P(t,s) + k'P(t,s+1) \tag{23}$$

where k is the probability of the transition from site $(S-1)$ to S per unit time, and k' that from $(S+1)$ to S. To find the principal solution to Eq. (23), we consider the initial condition of the walker at point m, i.e.,

$$P(0,s) = \delta_{s,m} \tag{24}$$

From Eqs. (23) and (24) the equation for the generating function $G(t,z) = \sum_{s=-\infty}^{\infty} z^s P(t,s)$ is

$$\frac{\partial G}{\partial t} = \left[kz - (k+k') + k'z^{-1}\right]G \tag{25}$$

$$G(t,1) = 1, \qquad G(0,z) = z^m$$

so that

$$G(t,z) = z^m \exp\left\{\left[kz - (k'+k) + k'z^{-1}\right]t\right\} \tag{26}$$

Noting the generating function for modified Bessel function[7] is

$$e^{(1/2)\tau(z+z^{-1})} = \sum_{s=-\infty}^{\infty} I_s(\tau) z^s \tag{27}$$

we have then from Eq. (26)

$$P(t,s) = \left(\frac{k}{k'}\right)^{(1/2)(s-m)} I_{s-m}\left[2t(kk')^{1/2}\right]e^{-(k+k')t} \tag{28}$$

Equation (21) is of the same form as Eq. (28) if we take

$$k \to \bar{\alpha}e^{-\bar{\beta}}$$

$$k' \to \bar{\alpha}e^{\bar{\beta}}$$

and

$$t \to \tau_p$$

To discuss a random walk with continuous time in detail, we need to know both the probability $\psi(t)$ for one step in time interval t and the transition probability matrix $\hat{M}$ for state transition in one step.[10] If we denote by $P(\nu_p, s)$ the probability for the walker to be at site s after ν_p steps, then we have in the matrix notation

$$\mathbf{P}(\nu_p + 1) = \hat{M}\mathbf{P}(\nu_p) \tag{29}$$

Therefore, for a given initial state $\mathbf{P}(0)$ the probability for the walker to be at some site at time t is

$$\mathbf{P}(t) = \sum_{n=0}^{\infty} [\psi(t)*]^n \left[1 - \int_0^t d\tau\, \psi(\tau)\right] \mathbf{P}(n)$$
$$= \sum_{n=0}^{\infty} [\psi(t)*]^n \left[1 - \int_0^t d\tau\, \psi(r)\right] \hat{M}^n \mathbf{P}(0) \tag{30}$$

where * indicates convolution operator. The Laplace transform is

$$\tilde{\mathbf{P}}(\lambda) = \left[\lambda - \frac{\lambda\tilde{\psi}(\lambda)}{1-\tilde{\psi}(\lambda)}(\hat{M}-1)\right]^{-1} \cdot \mathbf{P}(0) \tag{31}$$

where

$$\tilde{\psi}(\lambda) = \int_0^{\infty} e^{-\lambda t}\psi(t)\, dt \tag{32}$$

For a Poisson process[10] we have

$$\psi(t) = \tau_1^{-1} e^{-t/\tau_1} \tag{33}$$

then

$$\tilde{\psi}(\lambda) = (\tau_1\lambda + 1)^{-1} \tag{34}$$

Equations (31) and (34) yield

$$\tilde{\mathbf{P}}(\lambda) = \left[\lambda - \frac{1}{\tau_1}(\hat{M}-1)\right]^{-1} \mathbf{P}(0) \tag{35}$$

Thus,

$$\mathbf{P}(t) = \exp\left[\frac{1}{\tau_1}(\hat{M}-1)t\right]\mathbf{P}(0) \tag{36}$$

Comparing Eq. (36) with Eq. (11), one sees that Eq. (11) is equivalent to a random walk with $\hat{M} = (\hat{\mathcal{H}} + 1)\tau_1$ and $\psi(t)$ of Poisson form.

For a random walk with discrete time we have

$$\psi(t) = \delta(t - \tau_1)$$

and

$$\tilde{\psi}(\lambda) = e^{-\lambda\tau_1}$$

and then

$$\frac{\lambda\tilde{\psi}(\lambda)}{1-\tilde{\psi}(\lambda)} = \frac{\lambda e^{-\lambda\tau_1}}{1-e^{-\lambda\tau_1}} \xrightarrow{\lambda\tau_1 \ll 1} \frac{1}{\tau_1} \tag{37}$$

Hence, for large $t \gg \tau_1$ we have the same result as Eq. (36). We can take

$\tau_1 = \tau_m$ here. The transition probability for change $n \to n + \nu$ in ν_p steps for a random walk with discrete time is

$$W(n \to n+\nu) = \frac{\bar{\nu}_p!}{[(1/2)(\bar{\nu}_p+\nu)]!\,[(1/2)(\bar{\nu}_p-\nu)]!} \times \left(\frac{e^{-\bar{\beta}}}{2\cosh\bar{\beta}}\right)^{(1/2)(\bar{\nu}_p+\nu)} \left(\frac{e^{\bar{\beta}}}{2\cosh\bar{\beta}}\right)^{(1/2)(\bar{\nu}_p-\nu)} \tag{38}$$

$$\approx \frac{1}{(2\pi\bar{\nu}_p)^{1/2}} e^{-\nu^2/2\bar{\nu}_p} e^{-\nu\bar{\beta}} (\cosh\bar{\beta})^{-\bar{\nu}_p} \tag{39}$$

with

$$\bar{\nu}_p = \frac{\tau_p}{\tau_m} = 2\bar{\alpha}\tau_p \cosh\bar{\beta} \tag{40}$$

where we have used

$$\binom{N}{\frac{1}{2}(N-m)}\left(\frac{1}{2}\right)^N \to \frac{1}{(2\pi N)^{1/2}} \exp\left(-\frac{m^2}{2N}\right) \tag{41}$$

Expression (38) is normalized, but (39) is not. To normalize it, we rewrite (39) as

$$\frac{1}{(2\pi\bar{\nu}_p)^{1/2}} \exp\left[-\frac{1}{2\bar{\nu}_p}(\nu + \bar{\nu}_p\bar{\beta})^2\right]$$

$$= \frac{1}{(2\pi\bar{\nu}_p)^{1/2}} \exp\left(-\frac{\bar{\nu}_p}{2}\bar{\beta}^2\right) \exp\left(-\frac{\nu^2}{2\bar{\nu}_p}\right) \exp(-\nu\bar{\beta}) \tag{42}$$

Expression (42) is of Langer's form[1]:

$$W(n \to n+\nu) \propto \exp\left(-\frac{\nu^2}{2\bar{\nu}_p}\right) \exp\left\{-\frac{\beta}{2}[F(n+\nu) - F(n)]\right\} \tag{43}$$

as is (4.18) in Ref. 3, but differs from theirs by a normalization factor.

Let us now find the condition under which expression (42) is true. The generating functions for distributions (21), (38), and (42) are

$$G_M(\tau_p, z) = \exp\left[\left(\bar{\alpha}e^{-\bar{\beta}}z - 2\bar{\alpha}\cosh\bar{\beta} + \bar{\alpha}e^{\bar{\beta}}z^{-1}\right)\tau_p\right] \tag{44a}$$

$$G_B(\tau_p, z) = \left[\frac{e^{-\bar{\beta}}}{2\cosh\bar{\beta}} z + \frac{e^{\bar{\beta}}}{2\cosh\bar{\beta}} z^{-1}\right]^{\nu_p} \tag{44b}$$

$$G_G(\tau_p, z) = \exp\left[\frac{\bar{\nu}_p}{2}(\bar{\beta} - \ln z)^2 - \frac{\bar{\nu}_p}{2}\bar{\beta}^2\right] \tag{44c}$$

where subscripts M, B, and G indicate modified Bessel function, binomial, and Gaussian, respectively. In terms of Eq. (40), Eq. (44b) can be written as

$$\left[1+\left(\bar{\alpha}e^{-\bar{\beta}}z-2\bar{\alpha}\cosh\bar{\beta}+\bar{\alpha}e^{\bar{\beta}}z^{-1}\right)\tau_p\cdot\frac{1}{2\bar{\alpha}\tau_p\cosh\bar{\beta}}\right]^{2\bar{\alpha}\tau_p\cosh\bar{\beta}}$$

then for large $2\bar{\alpha}\tau_p\cosh\bar{\beta}$ hence large τ_p, (44b) approaches (44a).

In order to compare (42) with (21) we may calculate the moments for both. It is easy to verify that for Eq. (42)

$$\langle\nu\rangle_{\mathrm{G}}=-2\bar{\alpha}\bar{\beta}\tau_p\cosh\bar{\beta} \tag{45a}$$

$$(\langle\nu^2\rangle-\langle\nu\rangle^2)_{\mathrm{G}}=2\bar{\alpha}\tau_p\cosh\bar{\beta} \tag{45b}$$

By differentiating the generating function (44a) we have for the distribution (21)

$$\langle\nu\rangle_M=-2\bar{\alpha}\tau_p\sinh\bar{\beta} \tag{46a}$$

$$(\langle\nu^2\rangle-\langle\nu\rangle^2)_M=2\bar{\alpha}\tau_p\cosh\bar{\beta} \tag{46b}$$

While (45b) coincides with (46b), (45a) approaches (46a) only when $\bar{\beta}$ is very small. Under this condition we have from (44a)

$$G_M(\tau_p,z)\approx\exp\left[\frac{\bar{\nu}_p}{2}(z-2+z^{-1})\right] \tag{47}$$

On the other hand, from (44c) we have for large ν_p and small $\bar{\beta}$

$$G_{\mathrm{G}}(\tau_p,z)\approx\exp\left[\frac{\bar{\nu}_p}{2}(\ln z)^2\right]$$

$$\approx\exp\left[\frac{\bar{\nu}_p}{2}(z-2+z^{-1})\right] \tag{48}$$

because for $0\leqslant z\leqslant 1$ $z(\ln z)^2\sim(z-1)^2$, the leading term is the same as that of G_M. If we ignore the common factor $e^{-\nu\bar{\beta}}$ in (21) and (42), the nonvanishing higher moments for the Gaussian factor are

$$m_{2n}^{\tilde{\mathrm{G}}}=(2n-1)m_{2n-2}^{\tilde{\mathrm{G}}}\cdot m_2^{\tilde{\mathrm{G}}} \tag{49}$$

with

$$m_2^{\tilde{\mathrm{G}}}=2\bar{\alpha}\tau_p\cosh\bar{\beta}\approx 2\alpha\tau_p$$

and those for the distribution $I_\nu(2\bar{\alpha}\tau_p)$

$$m_{2n}^{\tilde{M}}=2\bar{\alpha}\tau_p\left[C_{2n-1}^1 m_{2n-2}^{\tilde{M}}+C_{2n-1}^3 m_{2n-4}^{\tilde{M}}+\cdots+C_{2n-1}^{2n-3}m_2^{\tilde{M}}\right] \tag{50}$$

with

$$m_2^{\tilde{M}} = 2\bar{\alpha}\tau_p$$

The leading terms are the same for both.

5. DISCUSSION

1. As one can see from the derivation in Section 3, it is not necessary to make an additional assumption that a Markoffian master equation exists on the phenomenological time scale. Metiu *el al.*'s result [(4.18) in Ref. 3] is an approximation of the result (21).

2. By calculating the first and second moments for a given transition probability, one can derive the Fokker–Planck equation.[11] From Eqs. (45a) and (45b) we have to order $\bar{\beta}$

$$\frac{\partial P}{\partial \bar{t}} = -\frac{\partial}{\partial \bar{n}}\left[\left(2\bar{\alpha}\bar{\beta}\tau_p \cosh \bar{\beta}\right)P\right] + \frac{1}{2}\frac{\partial^2}{\partial \bar{n}^2}\left[\left(2\bar{\alpha}\tau_p \cosh \bar{\beta}\right)P\right]$$

or

$$\frac{\partial P}{\partial \bar{t}} = \frac{\partial}{\partial \bar{n}}\left[\left(\bar{\alpha}\tau_p \cosh \bar{\beta}\right)\left(\frac{\partial \beta \bar{F}}{\partial \bar{n}}\right)P\right] + \frac{\partial^2}{\partial \bar{n}^2}\left[\left(\bar{\alpha}\tau_p \cosh \bar{\beta}\right)P\right] \qquad (51)$$

If we start from Eq. (9), we have similarly

$$\frac{\partial P}{\partial t} = \frac{\partial}{\delta n}\left[2\alpha \sinh\left(\frac{\beta \Delta F}{2}\right)P\right] + \frac{1}{2}\frac{\partial^2}{\partial n^2}\left[2\alpha \cosh\left(\frac{\beta \Delta F}{2}\right)P\right] \qquad (52)$$

For small $\beta \Delta F/2$, Eq. (52) can be written as

$$\frac{\partial P}{\partial t} = \frac{\partial}{\partial n}\left[\alpha \cosh\left(\frac{\beta \Delta F}{2}\right)\left(\frac{\partial \beta F}{\partial n}\right)P\right] + \frac{\partial^2}{\partial n^2}\left[\alpha \cosh\left(\frac{\beta \Delta F}{2}\right)P\right] \qquad (53)$$

One can recognize that Eq. (51) is just the coarse-grained form of Eq. (53).

Langer's form of the transition rate is not normalized, and leads to a diffusion constant in the F.P. equation equal to 1. The factor $\bar{\alpha}\tau_p \cosh \bar{\beta}$, which appears in both terms in the RHS of Eq. (51), contains the details of the heat bath and the coupling between spins and the heat bath. Only when $\bar{\alpha}\tau_p \cosh \bar{\beta}$ is slowly varying in the *coarse-grained* variable $\bar{n}$, can it then be moved out the differentiation and regarded as a time scaling factor. Then, Eq. (51) reduces to

$$\frac{\partial P}{\partial \tau} = \frac{\partial}{\partial \bar{n}}\left[\left(\frac{\partial \beta \bar{F}}{\partial \bar{n}}\right)P\right] + \frac{\partial^2 P}{\partial \bar{n}^2}$$

If this factor varies rapidly in $\bar{n}$, the effective potential would be

$$\bar{F} + \frac{1}{\beta}\ln\left(\bar{\alpha}\tau_p \cosh \bar{\beta}\right)$$

instead of $\bar{F}$. Phase transition would occur, for instance, even when $\bar{F}$ does not have a double-well structure at a temperature above the Curie point. In addition, even new critical points might occur.[12]

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